The implicit numerical method for the one-dimensional anomalous subdiffusion equation with a nonlinear source term

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Formulation of the problem

Consider the following subdiffusion equation with a nonlinear source term

$$ {}^{C}D^{\alpha}_{0+,t}U(x,t) = D_\alpha \frac{\partial^2 U(x,t)}{\partial x^2} + Q(x,t), \quad 0 \le x \le L, \quad t \ge 0, \tag{1}$$

supplemented with the boundary conditions

$$U(0,t) = f(t), \quad U(L,t) = g(t), \quad t \ge 0, \tag{2}$$

and initial condition

$$U(x,0) = h(x), \quad 0 \le x \le L, \tag{3}$$

where the generalized diffusion coefficient D_α is constant.

Numerical solution

Let $\Pi = \{(x,t) : x \in [0,l]; t \ge 0\}$ be a continuous region of solutions for the partial differential equation and $t = T$ will be the end time. Then the set $\bar{\Pi} = \{(x_i,t_j) \in \Pi : x_i = i\Delta x, i \in \{0,1,...,m\}, \Delta x = \frac{L}{m}; t_j = j\Delta t, j \in \{0,1,...,n\}; \Delta t = \frac{T}{n}\}$ we call the rectangular regular mesh described by the set of nodes.

Numerical scheme

$$\mathbf{A}\mathbf{U_k} = \mathbf{B},$$

$$\mathbf{A} = \begin{bmatrix} 1+2a & -a & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a & 1+2a & -a & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a & 1+2a & -a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -a & 1+2a & -a & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a & 1+2a & -a \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a & 1+2a \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} b_1 + aU_{0,k} \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{m-2} \\ b_{m-1} + aU_{m,k} \end{bmatrix}$$

$$a := \frac{D_\alpha w_{k,k}}{(\Delta x)^2},$$

$$b_i := U_{i,0} + \sum_{j=0}^{k-1} D_\alpha w_{j,k}(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}) + \sum_{j=0}^{k} w_{j,k} Q_{i,j},$$

$$w_{j,k} := \frac{(\Delta t)^\alpha}{\Gamma(2+\alpha)} \begin{cases} (\alpha+1-k)k^\alpha + (k-1)^{\alpha+1}, & j = 0 \\ (k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}, & 0 < j < k \\ 1, & j = k \end{cases}$$

Numerical example

Consider the first subdiffusion equation with a nonlinear source term

$$ {}^{C}D^{\alpha}_{0+,t}U(x,t) = \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{1}{6}\left(\frac{(x+1)(2x-3)t^{2-\alpha}}{\Gamma(3-\alpha)} - 2t^2 + 8\right), \tag{4}$$

supplemented with the boundary conditions

$$U(0,t) = -\frac{1}{4}(t-2)(t+2), \tag{5}$$

$$U(1,t) = -\frac{1}{6}(t-2)(t+2), \tag{6}$$

and initial condition

$$U(x,0) = -\frac{1}{3}(x+1)(2x-3). \tag{7}$$

The solution to the initial-boundary value problem (IBVP) given by Eq. (4-7) can be expressed by the function of two variables defined by the formula:

$$U(x,t) = \frac{1}{12}(t-2)(t+2)(x+1)(2x-3). \tag{8}$$

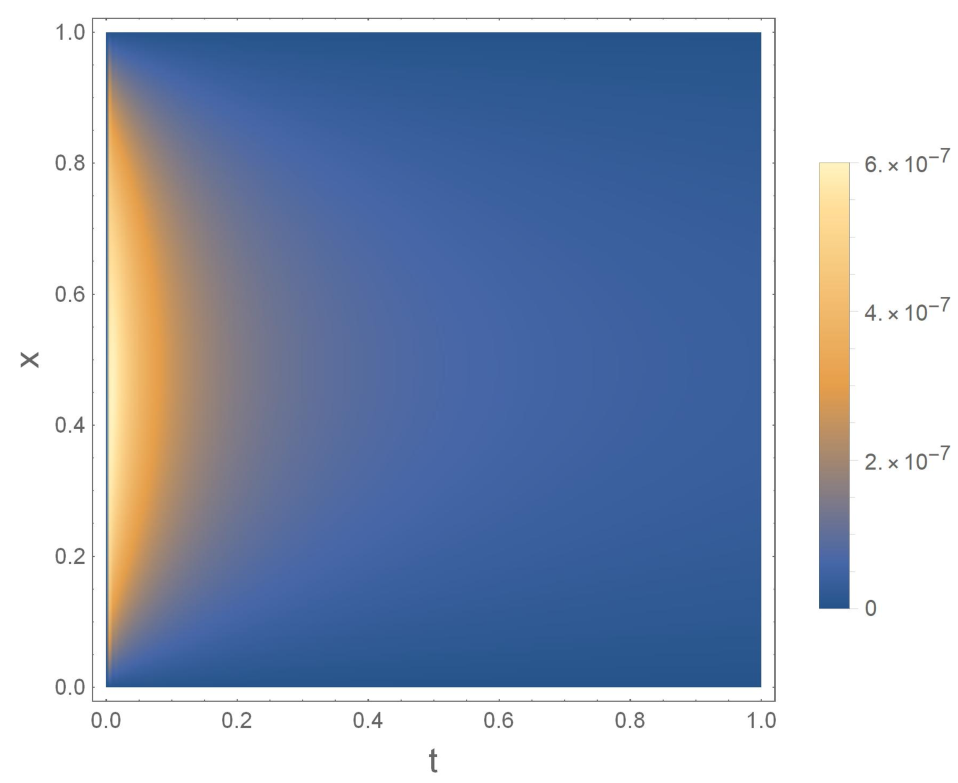

Figure 1: The absolute error generated by the numerical method for $\alpha = 0.75$, $\Delta x = \frac{1}{200}$, $\Delta t = \frac{1}{200}$.

Table 1: The average absolute error generated by the numerical method for $\alpha = 0.75$.

	$m = 25$	$m = 50$	$m = 100$	$m = 200$
$n = 25$	4.077e-6	4.162e-6	4.204e-6	4.226e-6
$n = 50$	1.083e-6	1.105e-6	1.116e-6	1.122e-6
$n = 100$	2.827e-7	2.886e-7	2.915e-7	2.93e-7
$n = 200$	7.304e-8	7.456e-8	7.532e-8	7.57e-8

Conclusions

The method proposed in the paper is an extension of the generalized Crank-Nicolson method for the one-dimensional sub-diffusion equation, where the source term is additionally taken into account. In the part of the method where the left-sided Riemann-Liouville integral was discretized, minor changes were made in the nodes numbering, which resulted in a more concise notation. The obtained numerical results are largely consistent with closed analytical solutions. The numerous simulations that were carried out did not indicate that the method was unstable in any of them. On the basis of the obtained results, it can also be concluded that the method is convergent. Let us point out that the method can be extended to the multidimensional case.

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